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PR-MATHS-01

Seat No. ________

M. Phil. (Sem. II) Examination

August - 2020

Mathematics : CMT - 20001

(Topology) (Old Course)

Time : **3** Hours] [Total Marks : **100**

Instructions : (1) All questions are compulsory.

(2) Each question carries 20 marks.

1 Answer following **ten** questions : **10×2=20**

(i) Define Basis for closed sets in a topological space (X, τ).

(ii) Define term : compact space. Give an example of a topological space is a compact space and give an example of a topological space not a compact space too.

(iii) Define Filter and Ultra Filter.

(iv) Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on *X*. Let $\mathcal{F} \to p$ in X, then prove that p is a cluster point of $\mathcal{F}$.

(v) Define ideal and maximal ideal.

(vi) Prove that, every single-ton set in $\mathbb{R}$ is zero set of $\mathbb{R}$.

(vii) Define C^* – embedded and C – embedded.

(viii) For a subspace *S* of a space *X*, prove that *S* is C – embedded in $X \Rightarrow$ it is C^* – embedded in *X*.

(ix) Define zero set nbhd and give an example of zero set nbhd.

(x) Define fixed ideal, free ideal, fixed *Z*–filter and free *Z*–filter.

2 Answer any **two** questions : **2×10=20**

(1) Prove that, countable intersection of zero sets in *X* is also a zero set of *X*.

(2) Let *X* be a topological space and *I* be an ideal of $C(X)$. Prove that $Z(I) = \{Z(f) / f \in I\}$ is a *Z*–filter on *X*.

(3) Prove that (a) $I \subseteq Z^{-1}(Z(I))$ and (b) $Z(Z^{-1}(\mathcal{F})) = \mathcal{F}$, where *I* is an ideal of $C(X)$ and $\mathcal{F}$ is a *Z*–filter on *X*.

3 Answer any **one** question : **1×20=20**

(1) Let I be a Z–ideal in $C(X)$. Prove that following statements are equivalent :

(i) I is a prime ideal.

(ii) I contains a prime ideal.

(iii) If $f, g \in C(X)$ and $fg = 0$, then either $f \in I$ or $g \in I$.

(iv) If $f \in C(X)$, there is $Z \in Z(I)$ such that f does not change sign on Z.

(2) For a topological space X, prove that following statements are equivalent :

(1) X is compact

(2) Every ideal in $C(X)$ is fixed

(3) Every maximal ideal in $C(X)$ is fixed

(4) Every ideal in $C^*(X)$ is fixed

(5) Every maximal ideal in $C^*(X)$ is fixed.

(3) Let (K, h) be a compactification of X and $h(X)$ is C^*–embedded in K. Prove that (K, h) is equivalent to $(\beta X, e)$.

4 Answer any **two** questions : **2×10=20**

(a) Prove that, every locally compact Hausdorff space is a Tychonoff space. What about the converse of this ? Justify your answer.

(b) Let (K_1, h_1) and (K_2, h_2) be two equivalent compactifications of X. Prove that there is homeomorphism $T : K_2 \to K_1$ such that $T o h_2 = h_1$ and $T^{-1} o\, h_1 = h_2$.

(c) Give an example of a C^*–embedded subspace, which is not C–embedded with required justification.

5 Answer any **two** questions : **2×10=20**

(1) Define Z–ideal. Let $\{I_\alpha / \alpha \in J\}$ be a family of Z–ideals in $C(X)$. Prove that $\bigcap_{\alpha \in J} I_\alpha$ is also a Z–ideal in $C(X)$.

(2) Let I be a Z–ideal in $C(X)$. Does I the Jacobson radical in (X)? Justify your answer.

(3) Let $I_k = \{f \in C(X) / f$ vanish outside of a compact subset of $X\}$. Prove that I_k is a free ideal in $C(X)$ if and only if X is locally compact space.

(4) Let X be a space and I be an ideal in $C(X)$. Prove that following statements are equivalent :

(i) $Z^{-1}(Z(I)) = I$

(ii) If $f \in C(X)$ and $Z(f) \in Z(I)$, then $f \in I$.

(iii) If $f \in C(X)$ and $Z(f) \in Z(g)$, for some $g \in I$, then $f \in I$.